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Euclidean Quantum Field Theory I: Equations for a Scalar Model

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EUCLIDEAN QUANTUM FIELD THEORY I:

Equations for a Scalar Model

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Abstract

The analytic continuations to imaginary time of the Green's functions of local quantum field theory define Euclidean Green's functions. Use of the proper-time method allows to represent these functions as multiple Wiener integrals of functionals that obey an infinite system of coupled integral equations which are similar to, and for the particular model of a complex scalar field in quadrilinear self-interaction considered here a limiting case of, systems studied in quantum statistical mechanics by Ginibre. As a consequence, the Euclidean Green's functions can for this model be obtained by a limiting process, with temperature and density going to infinity, from the reduced density matrices of a nonrelativistic Bose gas. Reduced functionals are defined and their equations determined as a preparatory step to renormalization in the superrenormalizable cases of two and three dimensions.

Introduction

It is well known[1] that quantum field theory in Minkowski space (MQFT), if a Lagrangian is given, can be cast in the form of an infinite system of coupled integral equations[2] for the infinite set of Green's functions. These systems of equations have so far been of little help except for studying certain formal properties of Green's functions (e.g. properties under gauge transformations in quantum electrodynamics[3] or how to define a Bethe-Salpeter kernel without recourse to perturbation theory[1,4,5]). The main obstacle to a nonformal use of those systems is our inaptitude to formulate properly the boundary conditions on such systems to make them mathematically meaningful. Prescriptions on how to break such systems off have been given at times but seem so far lacking in convincing justification as well as success.

One feature of those equations that doubtless increases the task is already the poor formulation of each single equation. E.g. in their momentum space form one encounters even under the most favorable of circumstances only conditionally convergent integrals, while in coordinate space they involve products of distributions. Dyson[6] has shown that this difficulty is overcome in perturbation theory by a rotation of the paths of integration. More generally, one may use simultaneous analytic continuation[7] of all functions in the equations and define the original functions by the boundary values of their continuations. Continuing to imaginary times respectively energies yields the Euclidean Green's functions studied in their own right by Schwinger[8] and Nakano[9]. These functions can be defined even without reference to a Lagrangian and may be associated with

a Euclidean quantum field theory (EQFT) whose characteristic symmetry group is not the Lorentz group but the orthogonal group in four dimensions.

EQFT is of no particular interest in an axiomatic framework[10], since the axioms are formulated directly in MQFT terms and all of EQFT is secondary. If, however, a Lagrangian is given, the situation is quite different. Then the investigation of the resulting particular system of integral equations for EQFT Green's functions becomes useful since the existence of a solution is a necessary condition for a corresponding MQFT to exist, provided one supposes the MQFT to possess a lowest-energy state as is done generally, and may be easier[11] to prove or disprove.

Compared to MQFT Green's functions systems the ones for EQFT functions have advantages: 1) The EQFT functions are singular only at coinciding arguments and do not have the light-cone singularities which are part of the origin of the ambiguities mentioned before. 2) Elliptic systems are more easily handled than hyperbolic systems. This expresses itself in the fact that for the analysis of EQFT relatively efficient mathematical tools are available, and this is the point of this series of papers. 3) There is an interesting and suggestive direct relation between EQFT and nonrelativistic quantum equilibrium statistical mechanics, which has no counterpart in MQFT. 4) The EQFT metric is always positive definite even if the MQFT metric, as in manifestly covariant quantum electrodynamics, is not.

Ultraviolet difficulties are the same in EQFT as in MQFT, but in the first case manifest themselves in terms of divergent, instead of meaningless, integrals. Renormalization of the coupled system of integral equations may be performed either with the help of limiting processes[12] or by renouncing manifest locality[13,5]. Both these ways are not suitable for our present purpose. We shall first construct a formal solution of the coupled system of EQFT integral equations and will invoke an ad hoc regularization of this solution wherever this seems to be illuminating. Our procedure is to derive from this formal solution new integral equations which are renormalized by eliminating the renormalization constants and which are the basis for a constructive existence proof for a nonformal solution.

EQFT and MQFT are on a comparable level as far as phenomena like spontaneous symmetry breakdown and vacuum degeneracy are concerned: about these one learns as much from EQFT as from MQFT Green's functions, although the question whether e.g. a symmetry breakdown holds for the S matrix also is directly answerable only on the basis of MQFT functions.

EQFT is clearly in a great disadvantage with respect to questions about observables, e.g. if there exists a particle interpretation and asymptotic completeness holds, what the scattering amplitudes are etc. However, as Lagrangian MQFT has resisted so far any attempt to extract from it such information (except in terms of most untrustworthy[14] perturbation expansions, or for models with scattering amplitudes identically zero) the indirect EQFT approach can be defended.

In Chapter 1 we show in a general way how EQFT Green's functions

are related to MQFT functions, and thereby derive some of their expected properties. This connection is, however, not needed for a study of EQFT itself.

In Chapter 2 we derive the starting equations for the model of a scalar complex field with quadrilinear self coupling and review at this example an operator formulation of EQFT given elsewhere[11].

In Chapter 3 we solve the equations formally, introduce an auxiliary intermediary field, and thereby obtain a form of solution which, if expanded, gives an expansion in increasing numbers of closed loops.

In Chapter 4 we obtain analogs of the Kirkwood-Salsburg and Mayer-Montroll integral equations[15,16] for distribution functions in classical statistical mechanics. The equations obtained closely resemble, and are for the model treated in this paper a limiting case of, equations used in nonrelativistic quantum statistical mechanics by Ginibre[17]. This is discussed in detail in Appendix A while in Chapter 4 itself the analogy to classical statistical mechanics is shown and exploited. Appendices B and C illustrate our equations and their properties in the lowest-dimensional cases of zero and one dimension, respectively, where no renormalization is needed.

In Chapter 5 we introduce, as preparatory to renormalization, reduced functionals, whereby in the superrenormalizable cases of two and three dimensions all terms that need be renormalized are collected in one simple equation. The renormalization of this equation by adaption and extension of a method due to Nelson[18] will be presented in the next

paper of this series, together with the closely related treatment of derivative couplings as occur in scalar and two-component spinor quantum electrodynamics.

1. Axiomatic Formulation of EQFT

For this chapter, we adopt the axiomatic approach to relativistic quantum field theory developed by Wightman.[19] We consider the theory of one hermitean scalar field $A(x)$ only.

Due to the stability of the vacuum (denoted by < and >), the spectrum condition, and their assumed temperedness as distributions, the vacuum expectation values

$$< A(x_o)A(x_1)\dots A(x_n) >, \quad x_i = (x_i^o, x_i^1, x_i^2, x_i^3)$$

of products of field operators are, as functions of $\xi_i = x_{i-1} - x_i$ $(i=1\dots n)$, boundary values of analytic functions $W_n\big((\zeta)\big)$, $(\zeta)=(\zeta_1\dots\zeta_n)$, with analyticity domain the tube

$$R_n = \{(\zeta) : \text{Im}\ \zeta_i \in V^+, \ \forall i\}$$

i.e. $\text{Im}\ \zeta_i^o > 0$, $(\text{Im}\ \zeta_i)^2 > 0$, with $g_{\mu\nu} = -\delta_{\mu\nu}(-1)^{\delta_{\mu o}}$. Due to relativistic invariance, $W_n\big((\zeta)\big)$ is analytic and single-valued in the extended tube

$$R_n' = \{ (\zeta) : \exists\ \Lambda_+(C),\ (\zeta) = (\Lambda_+(C)\zeta'),\ (\zeta') \in R_n \}$$

where $\Lambda_+(C)$ is a proper homogeneous complex Lorentz transformation:

$$\Lambda_+(C)^T g \Lambda_+(C) = g, \ \text{Det}\ \Lambda_+(C) = 1.$$

Due to local commutativity, $W_n\big((\zeta)\big)$ is analytic and single valued in the

permuted extended tube

$$R''_n = \bigcup_{\text{all } P} PR'_n$$

with

$$PR'_n = \{ (\zeta) : (\zeta) = (P\zeta'), (\zeta') \in R'_n \}$$

where $P \in S_{n+1}$ is a permutation

$$P : (0,1\ldots n) \rightarrow (P(0)P(1)\ldots P(n))$$

and if $\zeta_i = z_{i-1}-z_i$, then $P\zeta_i = z_{P(i-1)}-z_{P(i)}$. In R''_n, $W_n\big((P\zeta)\big) = W_n\big((\zeta)\big)$ due to our use of one field only. The Schwinger points

$$(\zeta_S) : \text{Re } \zeta^o_i = 0, \text{ Im } \zeta^{1,2,3}_i = 0, \forall_i$$

lie in the interior of R''_n if $\zeta_{s_{i+1}} + \ldots + \zeta_{s_k} \neq 0$ for all $1 \leq i+1 \leq k \leq n$. We may write

$$\zeta_{s_i} = x_{i-1}-x_i, \; x = (x^1,x^2,x^3,x^4) \text{ real}, \; x^4 = ix^o,$$

and introduce the Schwinger functions[8]

$$W_n\big((\zeta_s)\big) \equiv S(x_o x_1 \ldots x_n).$$

These Euclidean Green's functions are symmetric functions of n+1 4-vector arguments, invariant under the proper inhomogeneous orthogonal group in four dimensions (here called the Euclidean group), and real-analytic except at

points of coincidence of some arguments. (Their analytic continuations are invariant under the complex Euclidean group and are the original Wightman functions in different notation). They satisfy

(1.1) $$S(x_o \ldots x_n) = S(x_o^T \ldots x_n^T)^* = S(x_o^s \ldots x_n^s)^* = S(-x_o \ldots -x_n)$$

where $x^T = (x^1, x^2, x^3, -x^4)$, $x^s = (-x^1, -x^2, -x^3, x^4)$ and are, therefore, real if the theory is invariant under time reversal or space reflection.

The Green's functions

$$F(x_o \ldots x_n) \equiv \langle TA(x_o) \ldots A(x_n) \rangle$$

where T is the symbol for operator ordering with increasing times from right to left, are for noncoindicing arguments symmetric tempered Lorentz invariant distributions. Assuming that these functions can be extended[20] to such distributions for all arguments, Ruelle[21] has shown that the Fourier transforms

$$\tilde{F}(p_1 \ldots p_n) = \int dx_1 \ldots dx_n \, e^{i \sum x_i p_i} F(0 x_1 \ldots x_n)$$

are boundary values of analytic functions[22] which are invariant under the proper homogeneous Lorentz group. The Schwinger points (p_s) : Im $p_i^{1,2,3} = 0$, Re $p_i^o = 0$, $p_i^o = -p_i^4$, $\forall i$ lie inside the analyticity domain except for points where a nonempty partial sum of the vectors p_{s_i} vanishes. We shall write

$$i^n \tilde{F}((p_s)) \equiv \hat{S}(p_o p_1 \ldots p_n), \quad p_o = -p_1 - \ldots - p_n .$$

Then

$$(2\pi)^4 \, S(p_o \dots p_n)\,\delta(p_o + \dots + p_n) = \int dx_o \dots dx_n \, e^{-i\sum x_i p_i} S(x_o \dots x_n)$$

where $x_i p_i = x_i^1 p_i^1 + \dots + x_i^4 p_i^4$. If truncated Wightman functions[24] W^T and truncated Green's functions[21] F^T are introduced, the functions $\tilde{F}^T$ have no singularities at Schwinger points. Therefore, the functions $\tilde{S}^T(p_o \dots p_n) = \tilde{F}^T((p_s))$ are symmetric real-analytic functions, invariant under the homogeneous proper orthogonal group, and satisfy

$$(1.2) \qquad \tilde{S}^T(p_o \dots p_n) = \tilde{S}^T(p_o^T \dots p_n^T)^* = \tilde{S}^T(p_o^s \dots p_n^s)^* = \tilde{S}^T(-p_o \dots -p_n)$$

with definitions analogous to those in (1). They possess analytic continuations into the tube

$$(\mathrm{Im}\ p) \in D^m \equiv \bigcap_{\text{all } I} D_I^m \text{, with}$$

$$D_I^m = \{(\mathrm{Im}\ p) : \sum_{k=1}^{4} (\sum_{i \in I} \mathrm{Im}\ p_i^k)^2 < m^2\}$$

where I is a proper subset of {0, 1...n} and m > 0 is the lower bound of the mass spectrum (except for the vacuum) of the theory. It follows that, provided

$$\min_{i \neq j} (x_i = x_j)^2 > \varepsilon,$$

$$(1.3) \qquad \lim_{D \to \infty} S^T(x_o \dots x_n)\, e^{\sum_{i=o}^{n} \alpha_i x_i} = 0, \ (\alpha) \in D^m$$

where $\alpha_i x_i = \alpha_i^1 x_i^1 + \ldots + \alpha_i^4 x_i^4$, $D = (\max_{i,j}(x_i - x_j)^2)^{1/2}$. (3) shows the exponential decrease of $S^T(x_o \ldots x_n)$ for increasing distance between its arguments.

Having established the existence of Euclidean Green's functions in every theory that satisfies Wightman's postulates, we will further on proceed more heuristically, what seems justified as no physically nontrivial example of a Wightman theory is known. Our goal is, in fact, to construct models for which the axiomatic assumptions can be verified and on this basis perhaps be sharpened.

2. A Scalar Model

We consider the theory of one non-hermitian scalar field in d space-time dimensions corresponding to the Lagrangian density

$$(2.1)\quad L = \partial^\mu B^+ \partial_\mu B - m^2 B^+ B - \frac{1}{2} g\,(B^+ B)^2 + \alpha B^+ B$$

Here $\hbar = c = 1$, m is a finite mass that need not be the mass of a particle, g the positive coupling constant, and

$$(2.2)\quad \alpha = 2gG_o(0) + \delta m^2$$

where $G_o(0)$ a (for $d \geq 2$, infinite [25]) constant obtained from (2.6), and δm^2 another (for $d \geq 3$, negative infinite) constant determined in our next paper. The non-vanishing canonical commutators derived from (1) are

$$(2.3)\quad [B(\vec{x}, x^o), \dot{B}^+(\vec{x}\,', x^o)] = [B^+(\vec{x}, x^o), \dot{B}(\vec{x}\,', x^o)] = i\delta(\vec{x} - \vec{x}\,').$$

We denote the Euclidean Green's functions derived from

$$\langle T\, B(x_1)\ldots B(x_m) B^+(y_1)\ldots B^+(y_n)\rangle$$

as described in chapter 1 by $S(x_1\ldots x_m, y_1\ldots y_n)$ and their generating functional[26] by

$$(2.4)\quad S[\bar{J}, J] = \sum_{m=n=o}^{\infty} (m!\, n!)^{-1} \int \ldots \int dx_1\ldots dx_m dy_1\ldots dy_n \cdot$$

$$\cdot \bar{J}(x_1)\ldots\bar{J}(x_1) J(y_n)\ldots J(yn) S(x_1\ldots x_m, y_1\ldots y_n)$$

where $\bar{J}(x)$ and $J(x)$ are independent functions with algebraic meaning only.

One can show[11] that [27]

$$(2.5)\quad S[\bar{J}, J] = \langle T_\tau \int_{-\infty}^{+\infty} d\tau \int d\vec{x}\, [\bar{J}(\vec{x},\tau)B(\vec{x},\tau) + B^+(\vec{x},\tau)\; J(\vec{x},\tau)]\rangle$$

where

$$B(\vec{x},\tau) = e^{H\tau}B(\vec{x},0)e^{-H\tau}$$

$$B^+(\vec{x},\tau) = e^{H\tau}B^+(\vec{x},0)e^{-H\tau} \neq [B(\vec{x},\tau)]^+$$

and T_τ means ordering with increasing τ from right to left. (Note that the orthogonal-invariance of the left hand side of (5) is not manifest on the right.)

Field equations and canonical commutation relations to (1) give differential equations for the Green's functions, and analytic continuation results in the functional differential equations (we suppress the common argument x)

$$(2.6a)\quad (-\Delta + m^2)[\delta/(\delta J)]\; S + g\; [\delta^3/(\delta J^2\delta\bar{J})]\; S - \alpha\; [\delta/\delta J]\; S = \bar{J}\; S$$

$$(2.6b)\quad (-\Delta + m^2)\; [\delta/(\delta\bar{J})]\; S + g\; [\delta^3/(\delta J\delta\bar{J}^2)]\; S - \alpha\; [\delta/\delta\bar{J}]\; S = JS$$

where Δ is the Laplacian in d dimensions. Integrating (6) with the elliptic Green's function[28]

$$(2.7)\quad G_o\,(x - y) = (2\pi)^{-d} \int dk e^{ikx}\, (k^2 + m^2)^{-1}$$

(here and in the following we use the scalar product

$kx = \sum_{\alpha=1}^{d} k_\alpha x_\alpha$) gives vanishing boundary terms according to (1.3).

Interpreting J and $\bar{J}$ realistically as numerically-valued functions conjugate complex to each other, we may introduce the hermitian operator

$$H_\tau = H - \int d\vec{x}[\bar{J}(\vec{x},\tau)B(\vec{x},0) + B^+(\vec{x},0)J(\vec{x},\tau)]$$

where H is the canonical Hamiltonian to (1), adjusted such that $H \geq 0$, or, more generally, the time-displacement operator. Then

$$(2.8) \quad S[\bar{J},J] = \langle T_\tau \exp [-\int_{-\infty}^{+\infty} H_\tau d\tau]\rangle$$

and

$$|| T_\tau \exp [-\int_{-\infty}^{+\infty} H_\tau d\tau] || \leq \exp [-\int_{-\infty}^{+\infty} E_o(J_\tau)]$$

where $E_o(J_\tau)$ is the ground state energy to H_τ.

Using arguments based on functional integration, it can be shown[11] that (provided certain limiting processes are reasonably behaved as they are in renormalized perturbation theory) for two sets J_i and J_i' of k complex functions and k complex constants C_i

$$(2.9) \quad \sum_{i,j=1}^{k} \bar{C}_i C_j \; S[\bar{J}_i + \bar{J}_j', J_j + J_i'] \geq 0.$$

This property[29] allows [30] to give an operator formulation of EQFT, which for the present model takes the form: choose two pairs of canonically conjugate hermitian field operators in d dimensions $(i = 1, 2)$

$q_i(x)$, $p_i(x)$ such that

$$[q_i(x), q_j(x')] = [p_i(x), p_j(x')] = 0$$

$$[q_i(x), p_j(x')] = i\delta_{ij}\delta(x - x')$$

and (with summation convention) the Hamiltonian

$$(2.10) \quad H = \frac{1}{2}\int dx C_i^+(x) C_i(x)$$

where

$$C_i = p_i - \frac{1}{2}i\,(-\Delta + m^2)\,q_i - \frac{1}{4}\,igq_jq_jq_i + \frac{1}{2}i\alpha q_i.$$

Then, with $\rangle$ the state of lowest energy (which satisfies $C_i(x)\rangle = 0$), we have

$$(2.11) \quad S[\bar{J}, J] = \langle \exp\left(2^{-\frac{1}{2}}\int dx[\bar{J}(q_1 + iq_2) + J(q_1 - iq_2]\right)\rangle$$

as the generating functional of the equal-time ground-state expectation values of operator products. We have the momentum operators ($\alpha = 1...d$)

$$P_\alpha = \int dx p_i(x)\partial_\alpha q_i(x) = -\int dx q_i(x)\partial_\alpha p_i(x)$$

and (at least formally) the charge operator

$$Q = \int dx[q_1(x)p_2(x) - q_2(x)p_1(x)]$$

with the properties

$$[P_\alpha, q_i(x)] = -i\partial_\alpha q_i(x), \; [P_\alpha, p_i(x)] = -i\partial_\alpha p_i(x),$$

$$[Q, q_{1,2}(x)] = \pm iq_{2,1}(x), [Q, p_{1,2}(x)] = \pm ip_{2,1}(x),$$

$$[Q, H] = [Q, P_\alpha] = [H, P_\alpha] = [P_\alpha, P_\beta] = 0,$$

and the usual (Euclidean) invariance properties of the ground state.

This Hamiltonian theory[31] is of the general type studied by Araki[32], and the cluster property (1.3) (extended to two fields) finds here its natural place[33]. Due to absence of Lorentz invariance, locality holds only "non-relativistically". This theory is subjected to Haag's theorem[34] even for $d = 1$, when the corresponding "MQFT" describes only the anharmonic oscillator and thus is physically trivial. We shall discuss this theory in Appendix C.

3. Formal Solution and Loop Expansion

(2.6), integrated with (2.7), is formally solved by[35]

$$(3.1)\quad S[\bar{J}, J] = C \exp\left(-\frac{1}{2} g[\delta^4/(\delta J)^2(\delta\bar{J})^2] + \alpha[\delta^2/\delta J\delta\bar{J}]\right) \exp\left([\bar{J}G_o J]\right)$$

where we suppress the obvious integrations over d - dimensional space in the square-bracketed terms. C has to be chosen such that $S[0, 0] = 1$. We now use

$$(3.2)\quad \exp\left[-\frac{1}{2} g(\delta^2/\delta\psi^2)\right] \exp\left[(f\psi)\right]\Big|_{\psi = 0} = \exp\left[-\frac{1}{2}(f^2)\right]$$

to write (1) as

$$S[\bar{J}, J] = C \exp\left[-\frac{1}{2} g(\delta^2/\delta\psi^2)\right] \exp\left([(\psi + \alpha)\delta^2/(\delta\bar{J}\delta J)]\right) \exp\left([\bar{J}G_o J]\right)\Big|_{\psi = 0}$$

which can be evaluated[36] to be

$$(3.3)\quad S[\bar{J}, J] = C \exp\left(-\frac{1}{2} g[\delta^2/\delta\psi^2]\right)$$

$$\exp\left([\bar{J}(G_o^{-1} - \alpha - \psi)J] - \mathrm{Tr}\ \ln(1 - (\alpha + \psi)G_o)\right)\Big|_{\psi = 0}$$

In order to be able to use (2) again, we introduce the representations

$$(3.4a)\quad A^{-1} = \frac{1}{2}\int_0^\infty ds \ \exp\left(-\frac{1}{2} sA\right)$$

and

$$(3.4b)\quad \ln A - \ln B = \int_0^\infty s^{-1} ds[\exp\left(-\frac{1}{2} Bs\right) - \exp\left(-\frac{1}{2} As\right)]$$

where the parameter s is called "proper time"[37]. The solution that vanishes in infinity of the parabolic differential equation

$$(3.5a)\quad (\partial/\partial t)U(x,y,t) = [\frac{1}{2}\Delta - V(x,t)]U(x,y,t)$$

$$(3.5b)\quad U(x,y,+0) = \delta(x - y)$$

is the Wiener integral[38]

$$(3.6)\quad U(x,y,t) = \int P_{xy}^{t}(d\omega) \exp\left[-\int_0^t d\tau V(x(\tau), \tau)\right]$$

where $P_{xy}^{t}(d\omega)$ is the conditional Wiener measure on continuous paths $x(\tau)$ starting at $\tau = 0$ at y, ending at $\tau = t$ at x, and parametrized by ω.

$$(3.7)\quad \int P_{xy}^{t}(d\omega) = (2\pi t)^{-d/2} \exp\left[-\frac{1}{2} t^{-1}(x - y)^2\right]$$

is the fundamental solution of the heat equation[39] in d-dimensional space.

Expanding (3) and using (4), (6), and (2) we find
$S(x_1 \ldots x_m,\ y_1 \ldots y_n) = 0$ unless $m = n$, and

(3.8a) $$S(x_1 \ldots x_n,\ y_1 \ldots y_n) = \sum_{\Pi \varepsilon S_n} S(x_1 y_{\Pi(1)}, \ldots, x_n y_{\Pi(n)})$$

where

(3.8b) $$S(x_1 y_1, \ldots, x_n y_n) = 2^{-n} \int_o^\infty \cdots \int_o^\infty ds_1 \ldots ds_n$$

$$\exp\left(- \frac{1}{2} m^2 (s_1 + \ldots + s_n) \right) \int \ldots \int P^{s_1}_{x_1 y_1} (d\omega_1) \ldots P^{s_n}_{x_n y_n} (d\omega_n) \cdot$$

with

(3.9a) $$n(\omega_1 \ldots \omega_n) = C' \exp \left(- \frac{1}{2} g [\delta^2 / \delta\psi^2] \right)$$

$$\exp \left(\frac{1}{2} \alpha (s_1 + \ldots + s_n) + \sum_{i=1}^{n} \frac{1}{2} \int_o^{s_i} \psi(x_i(\sigma_i) d\sigma_i + \right.$$

$$+ \int_o^\infty t^{-1} dt \exp \left(- \frac{1}{2} m^2 t + \frac{1}{2} \alpha t \right) \int dz \int P^t_{zz} (d\omega)$$

$$\left. \cdot \exp \left[\frac{1}{2} \int_o^t \{ z(\tau) \} d\tau \right] \right) \Big|_\psi = 0$$

and by further expansion

(3.9b) $$n(\omega_1 \ldots \omega_n) = C' \sum_{\ell=0}^{\infty} (\ell!)^{-1} \prod_{j=1}^{\ell} \left(\int_o^\infty t_j^{-1} dt_j \cdot \right.$$

$$\left. \cdot \exp \left(- \frac{1}{2} m^2 t_j + \frac{1}{2} \alpha t_j \right) \int dz_j \int P^{t_j}_{z_j z_j} (d\bar{\omega}_j) \right).$$

$$\cdot \exp\left(\frac{1}{2}\alpha(s_1+\ldots+s_n) - \frac{g}{8}\sum_{i=1}^{n} \int_0^{s_i}\int_0^{s_i} d\sigma_i d\sigma_i' \delta\{x_i(\sigma_i)-x_i(\sigma_i')\} - \right.$$

$$- \frac{g}{4}\sum_{i<i'} \int_0^{s_i}\int_0^{s_{i'}} d\sigma_i d\sigma_{i'} \quad \delta\{x_i(\sigma_i)-x_{i'}(\sigma_{i'})\}$$

$$- \frac{g}{8}\sum_{j=1}^{\ell} \int_0^{t_j}\int_0^{t_j} d\tau_j d\tau_j' \delta\{z_j(\tau_j) - z_j(\tau_j')\} -$$

$$- \frac{g}{4}\sum_{j<j'} \int_0^{t_j}\int_0^{t_{j'}} d\tau_j \; d\tau_{j'} \delta\{z_j(\tau_j) - z_{j'}(\tau_{j'})\} -$$

$$\left. - \frac{g}{4}\sum_{i,j} \int_0^{s_i}\int_0^{t_j} d\sigma_i d\tau_j \delta\{x_i(\sigma_i) - z_j(\tau_j)\}\right).$$

The sum in (8a) goes over the $n!$ permutations of n elements, and C' differs from C by a factor and must be such that $n(\phi) = 1$. We shall refer to the sum in (9b) as the loop expansion (LE) and, combining in the last exponent the first two terms, abbreviate that exponent as

$$\Sigma(- \frac{1}{2} V_{ii} - V_{ii'} - \frac{1}{2} V_{jj} - V_{jj'} - V_{ij}) .$$

The interpretation of (8) with (9b) in terms of n open arcs and ℓ closed loops, which are contact-connected in themselves and with each other in all possible ways (if also the last exponential is expanded) is the same as given in quantum electrodynamics (QED) by Feynman[40]. While in the latter case the intermediary field is the electromagnetic one, here it is the field ψ that mediated contact interactions only and permitted us to obtain the combinatories of QED, which is simpler than that of the quadrilinear case.

The principal difference between (9b) and the expressions used by Feynman is that (9) contains the perfectly well-defined Wiener integral and thus allows a rigorous discussion, while the "integrals" used instead in QED or, as we could attempt here, in scalar MQFT (by letting in (8) and (9) $x_d \to ix^o$, $s \to i\,s$, $t \to i\,t$ etc.) are not integrals. Rather, they would be symbols associated with an exponential representation of the solution of the differential equation (Δ being here the Laplacian in $d-1$ dimensions)

$$(\partial/\partial t)U(x,y,t) = i[\tfrac{1}{2}\Delta - \tfrac{1}{2}\partial^2_{x^o} - V(x,t)]U(x,y,t)$$

with $U(x,y,o) = S(x-y)$, where for $V \equiv 0$

$$(3.10)\quad U(x,y,t) = i(2\pi it)^{-d/2} \exp\{i(2t)^{-1}[(\vec{x} - \vec{y})^2 - (x^o - y^o)^2]\}.$$

Since this fundamental solution is not positive, no measure can be obtained from it[41] nor useful estimates derived.

If instead of (1) we would have chosen the Lagrangian density for a hermitian scalar field

$$(3.11)\quad L = \tfrac{1}{2}[\partial^\mu A\partial_\mu A - m^2A^2 - \tfrac{1}{2}gA^4 + \alpha A^2]$$

with $\alpha = 3gG_o(0) + \delta m^2$, we would, under slight change of (8a) but none of (8b), have obtained (9) again with g replaced by $2g$ and a factor one-half for each t-integral. Thus, all equations we shall derive can be

immediately transcribed[35] into this case. In contrast, if e.g. a Lagrangian density for two scalar fields in trilinear interaction

$$(3.12)\quad L = \partial^\mu B^+ \partial_\mu B + \frac{1}{2}\partial^\mu A \partial_\mu A - M^2 B^+ B - \frac{1}{2} m^2 A^2 - gAB^+B + \alpha B^+ B + \frac{1}{2}\beta A^2 + \gamma A$$

is chosen, for the Green's functions that do not contain the A field similar formulas as before are obtained, with the replacements

$$-\frac{g}{8}\int_o^s\int_o^s d\sigma d\sigma' \delta\{x(\sigma) - x(\sigma')\} \rightarrow \frac{1}{2} g^2 \int_o^s\int_o^s d\sigma d\sigma' G_o\{x(\sigma) - x(\sigma')\}$$

etc., i.e. instead of a positive contact interaction we obtain a negative singular finite-range interaction. We show in appendix A that this excludes at least the possibility of obtaining such EQFT by a limiting process from nonrelativistic quantum statistical mechanics (QSM) as described there. In fact, the MQFT to (12) is suspected[42] (and for $d = 2$ proved[43]) not to possess a translation invariant lowest-energy state[44], which also deprives the formally corresponding EQFT of its basis[45], since for the transition from MQFT to EQFT the spectral condition[10] is crucial.

The LE behaves very differently from perturbation theory obtained by expanding (1). As given in (9b) the LE has no meaning yet[46]. If, however, we introduce an ad hoc regularization that a) replaces the delta functions by smooth integrable functions, b) replaces[47] the lower limits zero of the s- and t- integrals by $\varepsilon>0$, c) replaces the

upper limits by $E<\infty$, and d) replaces the infinite z-integration volume by a finite one, then the LE converges like an exponential series since all terms are then trivally majorized. In contrast, the usual perturbation expansion does not converge even under such drastic modifications[11] of the model.

The modifications just described give $n(\omega_1 \ldots \omega_n)>0$. If, upon gradual removal of the modifications, the functions $n(\omega_1 \ldots \omega_n)$ approach the ones of the unmodified model, $n(\omega_1 \ldots \omega_n) \geqslant 0$ must hold for these and because of (8) for $S(x_1 .. x_n, y_1 .. y_n)$ also. That such approach takes place is probable on the basis of earlier results[11] and the (at least in perturbation theory) known insensitivity of renormalizable theories against the manner of regularization. Another class of regularizations applicable to the present model is that described in Appendix A and gives the same result $n(\omega_1 \ldots \omega_n) \geqslant 0$.

As the space-time volume goes to infinity, C' becomes infinite (or zero, depending on the details of the regularization) since it depends on that volume exponentially[11]. The problem to show that the $n(\omega_1 \ldots \omega_n)$ have limits is, however, essentially the same as to show in classical statistical mechanics that the thermodynamical limit exists for distribution functions. The classical methods can, therefore, be applied.

4. Kirkwood - Salsburg and Mayer - Montroll Integral Equations

We single out in (3.9a) the path ω_1 from the others. Using, with the abbreviations introduced after (3.9b),

$$\exp\left(-\frac{1}{2}g[\delta^2/(\delta\psi)^2] \quad \exp\frac{1}{2}\int_o^{s_1} d\sigma_1\psi[x_1(\sigma_1)]\right) =$$

$$= \exp\left(\frac{1}{2}\int_o^{s_1} d\sigma_1\psi[x_1(\sigma_1)]\right) \exp\left(-\frac{1}{2}g\,[\delta^2/(\delta\psi)^2] -\right.$$

$$\left.-\frac{1}{2}g\int_o^{s_1} d\sigma_1\delta/\delta\psi[x_1(\sigma_1)] - \frac{1}{2}V_{11}\right),$$

$$\exp\left(-\frac{1}{2}g\int_o^{s_1} d\sigma_1\delta/\delta\psi[x_1(\sigma_1)]\right) \exp\left(\frac{1}{2}\int_o^{s_i} d\sigma_i\psi[x_i(\sigma_i)]\right) =$$

$$= \exp\left(\frac{1}{2}\int_o^{s_i} d\sigma_i\psi[x_i(\sigma_i)] - V_{1i}\right),$$

and

$$\exp\left(-\frac{1}{2}g\int_o^{s_1} d\sigma_1\delta/\delta\psi[x_1(\sigma_1)]\right) \exp\left(\frac{1}{2}\int_o^{t} d\tau\psi[z(\tau)]\right) =$$

$$= [1 - K(\omega_1,\bar{\omega})] \quad \exp\left(-\frac{1}{2}g\int_o^{s_1} d\sigma_1\delta/\delta\psi[x_1(\sigma_1)]\right),$$

where $K(\omega,\bar{\omega})$ is the "bond" functional

$$K(\omega,\bar{\omega}) = 1 - \exp\left(-\frac{g}{4}\int_o^{s} d\sigma\int_o^{t} d\tau\,\delta[x(\sigma)-z(\tau)]\right), \tag{4.1}$$

we find, by expanding in powers of the bond functional and comparing with the definition (3.9a) of $n(\omega_1 \ldots \omega_n)$, with $n(\emptyset) = 1$ and empty products being one,

$$(4.2)\quad n(\ _1 \ldots\ _n) = \exp\left(-\frac{1}{2}V_{11} - \sum_{i=2}^{n} V_{1i}\right)\cdot$$

$$\cdot\sum_{\ell=0}^{\infty}(-1)^{\ell}(\ell!)^{-1}\prod_{j=1}^{\ell}\left[\int_0^{\infty} t_j^{-1}\,dt_j\, e^{-\frac{1}{2}m^2 t_j}\int dz_j \int P^{t_j}_{z_j z_j}(d\bar{\omega}j)\cdot\right.$$

$$\left.\cdot K(\omega_1,\bar{\omega}j)\right]\, n(\omega_2\ldots\omega_n\bar{\omega}_1\ldots\bar{\omega}_\ell)$$

the analog of the equation of Kirkwood and Salsburg (KS)[15] for distribution functions in the grand canonical ensemble of classical statistical mechanics (CSM).

Either iterating (2) n-1 times or, more conveniently, dealing with all trajectories $\omega_1 \ldots \omega_n$ as done above with ω_1, we obtain the analog of the equation of Mayer and Montroll (MM)[16]

$$(4.3)\quad n(\omega_1\ldots\omega_n) = \exp\left[-\frac{1}{2}\sum_{i=1}^{n} V_{ii} - \sum_{i<i} V_{ii'}\right]\cdot$$

$$\cdot\sum_{\ell=0}^{\infty}(-1)^{\ell}(\ell!)^{-1}\prod_{j=1}^{\ell}\left[\int_0^{\infty} t_j^{-1}dt_j\, e^{-\frac{1}{2}m^2 t_j}\int dz_j \int P^{t}_{z_j z_j}(d\bar{\omega}_j)\cdot\right.$$

$$\left.\cdot K(\omega_1\ldots\omega_n,\bar{\omega}_j)\right] n(\bar{\omega}_1\ldots\bar{\omega}_\ell)$$

we find, by expanding in [illegible]

with the definition [illegible]

empty products being [illegible]

(4.2) $n($ [illegible]

$+ \sum_{\ell=0}^{\infty} (-1)^{\ell} (\ell!)^{-1}$ [illegible]

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(4.3) $n(\omega_i$ [illegible]

$+ \sum_{\ell=0}^{\infty} (-1)^{\ell} (\ell!$ [illegible]

where

$$(4.4)\quad K(\omega_1 \ldots \omega_n, \bar{\omega}_j) = 1 - \exp\left[-\sum_{i=1}^{n} V_{ij}\right].$$

The relation of (2) and (3) to equations derived in QSM by Ginibre[17] is discussed in Appendix A. Here we will rather draw a parallel to CSM. In (2) and (3) the Wiener integration is partly redundant.[48] We may define

$$(4.5a)\quad \int_0^\infty t^{-1} dt\, e^{-\frac{1}{2} m^2 t} \int dz \int P_{zz}^t(d\omega).$$

$$\cdot F\left\{\int_0^t d\tau\,\delta[\cdot - z(\tau)]\right\} \equiv \int Q\,(db)\, F\,\{b\}$$

where F{b} is a functional defined in the space of non-negative integrable functions [49]

$$(4.5b)\quad b(z') = \int_0^t d\tau\,\delta\left(z' - z(\tau)\right)$$

which are such that $b(z')dz'$ is the time the Brownian particle executing the closed-path motion $z(\tau)$ spends in the volume dz', and Q{db} is a (nonfinite[50]) measure of that space. We will refer to the "blob" picture and the "blob" measure. The functionals n(..) depend only on blobs and may be written as functionals $\bar{n}(b_1 \ldots b_n)$, or briefly n(1...n), of blobs, since

$$(4.5c)\quad s_i = \int dx b_i(x) \equiv \| b_i \|, \quad V_{ij} = \frac{g}{4} \int dz b_i(z) b_j(z)$$

i.e. the "interaction potential" between blobs is determined by the degree

of overlap. If we consider a blob as an internal degree of freedom of a particle, (2) and (3) become (except for the self-potential of a blob) identical with the KS and MM equations of CSM, with the integration over the internal degree of freedom.

We shall find in the next paper that renormalization can be simply expressed only in the Wiener picture (although the final formulae can be transcribed into the blob picture) since a divergence arises only if the Brownian particle stays in the environment of a point instead of only returning to it at a later time. However, the blob picture is sometimes convenient and is the basis of our discussing (2) and (3) in the following in terms of CSM.

The derivation of the KS and MM equations from the modified LE described in Chapter 3 can be criticized on the following counts: a) In QFT the volume is intrinsically infinite, b) The ad hoc regularizations are not justified but (presumedly) correctly renormalized MQFT should be started from, c) The infinite sums in (2) and (3) may not converge absolutely after partial (or later full) removal of the modifications, d) (3.1) and (3.9) are formally summed perturbation theory and, therefore, not a satisfactory starting point if non-perturbation theoretical phenomena, like symmetry breakdown or bound states[52], are to be expected.

a) and b) cannot be answered convincingly at this stage but one may argue: a) The infinite-volume problem is present in CSM in reverse: actual systems are finite. Nevertheless, condensation,

of overlap [illegible]

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identical with the [illegible] of QCD, [illegible]

over the intense [illegible]

We shall [illegible] expect [illegible]

simply expression [illegible]

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picture in [illegible]

and (3) in [illegible]

The derivation [illegible]

described in [illegible]

a) In QFT [illegible]

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mathematically possible only in an infinite volume, is observable in finite containers. Thus, what is observable does not depend qualitatively on the size of the system (provided it is large enough), except for the following: in strictly infinite space, the observer, at a fixed point, will (with probability one) stay forever either in gas or in liquid even if both coexist and any convex linear combinations of the distribution functions solves the KS equations. b) It is possible to build more familiar regularizations into the equations. This requires to introduce non-continuous paths, however, since the continuity of paths is consistent only with the Wiener measure and thus leads to $G_o(0)$ - divergence due to "tadpoles". It is not obvious how to extend the renormalization procedure that will be given later to other cases. The same applies to renormalization by limiting processes in field equations themselves[12]. Therefore we shall be content to examine a constructive solution of the renormalized equations on its relation to renormalized EQFT[53] later. c) This must be checked later and, if necessary, a summation prescription be given and justified[54]. In CSM, the corresponding summation is not considered a problem: often a hard-core potential (for which perturbation theory is meaningless altogether) of strictly finite range is assumed or at least considered[55] for almost all phenomena a permissible approximation, and then the sums are finite. d) The last remark is relevant also here: the finite-sum KS equations are (at least for finite volume) rigorous[56] and unproblematic, and nevertheless expected[55] to describe phase transitions like condensation

mathematically possible [illegible]

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NS equatio[illegible]

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and crystallization i.e. have in the latter case a "broken-symmetry" solution. Thus, non-invariant[57] and, more generally, non-perturbative behaviour of solutions is not incompatible with an invariance an equation may possess, as was first proposed in field theoretical context by Heisenberg and is now generally accepted. Of course, the attempted method of solution must not exclude such behaviour.

In the CSM interpretation of (2) and (3) the coupling constant g stands at the place of $\beta = (kT)^{-1}$, and $-m^2$ plays a role comparable to that of the chemical potential. Thus, from results on the analytic properties of distribution functions in CSM[58] one infers analyticity in both these variables, especially, for fixed real m^2 analyticity in g in an environment of the positive real axis. This inference will be verified for the simple cases $d = 0$ and $d = 1$ in Appendices B and C, respectively.

(2) and (3) may be written

(4.6) $N = N_o + \mathrm{Op}\, N$

where N is the vector $\left(n(\omega_1),\, n(\omega_1,\omega_2),\ldots\right)$ or $\left(n(1),\, n(12),\ldots\right)$,

Op a (Wiener resp. blob) integral operator, and N_o the contribution from $n(\emptyset) = 1$, written separately. Simple estimates now show that under the modifications described in Chapter 3, but with infinite instead of finite space-time, (2) and (3), or (6), have in a suitable Banach space unique solutions that can be obtained from the inhomogenous term

of (6) by iteration. Since the technique hereto is also implicit in Appendix C, we will not discuss this further.

For $d = 0$ and $d = 1$, α is to be chosen finite and the KS or MM equations can be used as they stand. This is done in Appendices B and C For $d = 2$ and $d = 3$, the only divergence expected is that α will have to be infinite as (2.2) suggests. For $d = 4$, also amplitude- and coupling-constant renormalization will be necessary. Since these renormalizations are more complicated, we confine ourselves now to $d = 3$ where renormalization leads to equations that are relatively simple and can therfore be discussed conclusively. This discussion will be given in a later paper of this series.

5. Reduced Functionals

The primitively divergent diagrams for $d = 3$ are shown in Fig. 1, together with the diagrams that they separate into by our introduction of the intermediary ψ - field, whose lines are broken. Inspection of (2) and (3) shows that the contributions A and C, as far as they arise anew on the right hand sides, are isolated by introducing "reduced" functionals that do not contain factors corresponding to bare arcs. Since we have to allow for reducible diagrams also, we introduce an unknown functional $f(\omega)$ of one trajectory and define (arguments crossed out are omitted)

$$(5.1a)\quad n_r(\omega_1\ldots\omega_n) = n(\omega_1\ldots\omega_n) -$$

$$- \sum_i f(\omega_i) n(\omega_1 .. \not\omega_i .. \omega_n) + \sum_{i<i'} f(\omega_i)\, f(\omega_{i'}) n(\omega_1 .. \not\omega_i ... \not\omega_{i'} .. \omega_n) - + ..$$

whence

$$(5.1b)\quad n(\omega_1 .. \omega_n) = n_r(\omega_1\ldots\omega_n) + \sum_i f(\omega_i) n_r(\omega_1 \ldots \not\omega_i .. \omega_n) + ..$$

$$+ \sum_{i<i'} f(\omega_i) f(\omega_{i'}) n(\omega_1 .. \not\omega_i .. \not\omega_{i'} .. \omega_n) + ..\, .$$

Solving (4.2) and (4.3) for n_r requires lengthy formulas. Therefore, we take advantage of the blob picture and introduce the generating functional of functionals of blobs, the expansion element being a general functional $J(b)$ of a blob,

$$N\{J\} = \sum_{n=o}^{\infty} (n!)^{-1} \int .. \int Q(db_1)..Q(db_n) J(b_1)..J(b_n)$$

$$\exp\left(-\frac{1}{2}\sum_i V_{ii} - \sum_{i<i'} V_{ii'}\right).$$

The generating functional of the n - functionals is, according to (3.9b),

$n\{J\} = N\{1\}^{-1} N\{J + 1\}$. The KS equation for the N - functional is (the index b at N and n denotes functional differentiation $\delta/\delta J(b)$)

$$N_b\{J\} = e^{-\frac{1}{2} V_{bb}}\, N\{e^{-V_{b\cdot}}\, J(\cdot)\}$$

such that for the n - functional it is

$$(5.2)\qquad n_b\{J\} = e^{-\frac{1}{2}V_{bb}}\, n\{e^{-V_b}J(\cdot) + (e^{-V_{b\cdot}} - 1)\}$$

while the MM equations are

$$N_{1\cdots n}\{J\} = \exp\left(-\frac{1}{2}\sum_{i=1}^{n} V_{ii} - \sum_{i<i'} V_{ii'}\right) N\{e^{-\sum_{i=1}^{n} V_{i\cdot}} J(\cdot)\}$$

and therefrom

$$(5.3)\qquad n(1\ldots n) = \exp\left(-\frac{1}{2}\sum_{i=1}^{n} V_{ii} - \sum_{i<i'} V_{ii'}\right) n\{(e^{-\sum_{i=1}^{n} V_i} - 1)\}.$$

The definition $n_r\{J\} = \exp\{-\int Q(db)J(b)f(b)\}\, n\{J\}$

gives a rather unsymmetric KS equation for n_r, while the MM equation remains manifestly symmetric:

$$(5.4a)\qquad n_r(1..n) + \sum_{i=1}^{n} f(i) n_r(1..\cancel{i}..n) + \ldots =$$

$$= \exp\{-\frac{1}{2}\sum_{i=1}^{n} V_{ii} - \sum_{i<i'} V_{ii'} + \int Q(db)f(b)(e^{-\sum_{i=1}^{n} V_{ib}} - 1)\}\cdot$$

$$\cdot n\{(e^{-\sum_{i=1}^{n} V_{i\cdot}} - 1)\} \equiv s(1..n)$$

such that

$$(5.4b)\qquad n_r(1..n) = s(1..n) - \sum_{i=1}^{n} f(i)\, s(1..\cancel{i}..n) \pm \ldots$$

We now define

$$(5.5)\quad \rho(1..n) \equiv n_r(1...n)f(1)^{-1}..f(n)^{-1},$$

and with

$$(5.6)\quad \rho(1...n) = \sigma(1...n) - \sum_{i=1}^{n} \sigma(1..i..n) + \sum_{i<i'} \sigma(1..\cancel{i}..\cancel{i}'..n) - + \cdot\cdot$$

it is seen that the following two choices of f are convenient:

$$(5.7a)\quad f(b) = \exp\left(- \frac{1}{2} V_{bb} - \int Q(db')K(b,b')\right) \equiv 1 + \varepsilon(b)$$

which leads to

$$(5.7b)\quad \sigma(1...n) = \exp\left(- \sum_{i<i'} V_{ii'} - \int Q(db)[\varepsilon(b)K(b_1..b_n,b) - \sum_{i=1}^{n} K(b_i,b) + \right.$$

$$\left. + K(b_1...b_n,b)]\right) \rho\left(- K(b_1..b_n,\cdot)f(\cdot)\right)$$

and

$$(5.8a)\quad f(b) = \exp\left(- \frac{1}{2} V_{bb} - \int Q(db')K(b,b')f(b')\right)$$

which leads to

$$(5.8b)\quad \sigma(1..n) = \exp\left(- \sum_{i<i'} V_{ii'} + \int Q(db)[\sum_{i=1}^{n} K(b_i,b) - \right.$$

$$\left. - K(b_1...b_n,b)]f(b)\right) \rho\left(- K(b_1...b_n, \ )f(\cdot)\right)$$

where the K are the blob-transcriptions of (4.1) and (4.4) and $\rho\left(\ ... \right)$ is the generating functional of the ρ-functionals.

In both cases self-interaction and inter-blob interaction are separated; (7) and (8) differ only in the manner in which higher corrections that do not matter for renormalization are distributed. While (7a) gives f explicitly for use in (7b) with (6), (8a) is an integral equation in blob space of the Hammerstein type. No self-consistency problem is here involved, however, since (8a) is likely always to have a solution if (7a) is finite. Moreover, there is no such problem in the separation (7) and no consistency problem is expected physically.

We note that[59]

$$(5.9) \qquad 0 \leq \sum_{i=1}^{n} K(b_i,b) - K(b_1..b_n,b) \leq \sum_{i<i'} K(b_i,b)K(b_{i'},b)$$

such that, because of, effectively, $K(b_i,b) = 0(\|b_i\| \, \|b\|)$

the integrals in the exponents in (7b) and (8b) will be found to converge (for $d = 3$)[50] due to (3.7) and (4.5) if

$$(5.10) \qquad \varepsilon(b) = 0(\|b\|).$$

The same is true for the integrals implicit in the last factors of (7.b) and (8.b) if, for orientation, one inserts in (6) for σ (with $\sigma(\emptyset) = 1$ always) the first approximation

$$\sigma(1...n) = \exp\left(-\sum_{i<i'} V_{ii'}\right), \quad \sigma(1) = 1.$$

This shows that if (7a) turns out to be finite with (10) satisfied, the present renormalization is likely to have been successful for $d = 3$ but to be insufficient for $d = 4$ where more than mass renormalization is required.

This shows [illegible]

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$\mu = 3$ for [illegible]

renormalize [illegible]

Appendix A. Relation to Quantum Statistical Mechanics

The connection between EQFT of charged scalar particles and QSM of nonrelativistic neutral Bose particles[60] is best established by comparing (3.8) and (4.2) with G(5.4) and G(6.8), G(6.9) of Ginibre. The QSM combinatorics is dealt with simpler, however, by using for the generating functional of reduced density matrices

(A.1) $$Z_\beta\{\bar{J},J\} = \sum_{n=o}^{\infty} (n!)^{-2} \int..\int dx_1..dx_n\, dy_1..dy_n$$

$$J(x_1)..J(x_n)J(y_1)..J(y_n)\ \rho_\beta(x_1..x_n,y_1..y_n)$$

the expression

(A.2) $$Z_\beta\{\bar{J},J\} = C\ \exp\{-2\int_o^\beta d\tau \iint dxdy\ V(x-y)\cdot$$

$$\delta^2/\left(\delta\psi(x,\tau)\ \delta\psi(y,\tau)\right)\}\ \ \exp[JK(1-K)^{-1}J\text{-}\mathrm{Tr}J n(1-K)]$$

where K is the integral operator[61]

(A.3) $$K(x,y)=z\int P^\beta_{xy}(d\omega)\ \exp[\tfrac{1}{2}\beta V(0) + \tfrac{1}{2}\int_o^\beta d\tau\psi(x(\tau),\tau)]$$

and C such that $Z_\beta\{0,0\}=1$. Expanding the last exponential in (2) in powers of K and using (3.2) gives G(5.4) with substitution of G(5.8).

(A.2) is related to (3.3) as follows. If we let

(A.4) $$V(x - y) = \frac{g}{4}\,\beta\,\delta^\beta_{reg}(x - y)$$

where δ^{β}_{reg} is a β-dependent, regularized delta function with $\delta^{\beta}_{reg} \to \delta$ as $\beta \to 0$, and imagine β to become very small, the implicit τ - dependence of ψ in (3) may be neglected, and the explicit τ - dependence has a trivial effect such that it is then equivalent to write the last exponent in (3) as $2^{-1}[\psi(x)+\psi(y)]$ and the first in (2) as $2^{-1}g[\delta^2/\delta\psi^2]$.
with $z = \exp \beta\mu$ and keeping μ finite, we have for very small β

(A.5a)
$$K(1-K)^{-1} \approx \sum_{s=1}^{\infty} \int P^{\beta s}_{xy}\,(d\omega)\ \exp[\beta\mu s + $$
$$+ \frac{1}{2}\, g\ \beta^2 \delta^{\beta}_{reg}(0)\ s + \frac{1}{2} \int_0^{\beta s} d\sigma\psi\big(x(\sigma)\big) \approx$$
$$\approx \beta^{-1} \int_0^{\infty} ds \int P^{s}_{xy}(d\omega)\ \exp[-\frac{1}{2}m^2 s + \frac{1}{2}\alpha s + \frac{1}{2}\int_0^{s} d\sigma\psi\big(x(\sigma)\big)]$$

and

(A.5b)
$$-\operatorname{Tr}|n\ (1-K) \approx \sum_{t=1}^{\infty} t^{-1} \int dz \int P^{\beta t}_{zz}(dw)$$
$$\exp[\beta\mu t + \frac{1}{g} g\beta^2 \delta^{\beta}_{reg}(0) t + \frac{1}{2} \int_0^{\beta t} d\tau\psi\big(x(\tau)\big)] \approx$$
$$\approx \int_0^{\infty} t^{-1} dt \int dz \int P^{t}_{zz}(d\omega)\ \exp[-\frac{1}{2}m^2 t + \frac{1}{2}\,\alpha t + \frac{1}{2}\int_0^{t} d\tau\psi\big(z(\tau)\big)]$$

provided we set

(A.6)
$$\mu = -\frac{1}{2}\, m^2 + \frac{1}{2}\,\alpha - \frac{g}{8}\,\beta\delta^{\ \beta}_{reg}(0)$$

which, however, is meaningless as it stands and not generally correct even for $d = 1$ (see Appendix C).

Actually, the "approximation" (5) mainly shows that the combinatorics is correct but not what a suitable sequence of values for μ is as $\beta \to 0$. Such sequence can be found by comparing (5.7a) and (5.8a) with their QSM analogs (which resemble (4-12) of the last paper of ref. 17). This will be done in our next paper.

However, comparison of (5) with (3.8) and (3.9b) gives, for a suitable choice of μ as function of β,

$$\text{(A.7)} \quad \lim_{\beta\to 0} 2^{-n}\beta^n \rho_\beta(x_1 \ldots x_n, y_1 \ldots y_n) = S(x_1 \ldots x_n, y_1 \ldots y_n)$$

provided $M = 1$ and the potential is chosen as in (4). More generally, (4), (6) and (7) would read

$$\text{(A.4')} \quad V(x-y) = (4M^2)^{-1} g\beta\hbar^2 \delta^{\beta}_{reg}(x-y)$$

$$\text{(A.6')} \quad \mu = -(2M)^{-1} m^2 c^2 + \Delta\mu(\beta)$$

$$\text{(A.7')} \quad \lim_{\beta\to 0} [\beta\hbar^3/(2M)]^n \rho_\beta(x_1 \ldots x_n, y_1 \ldots y_n) = S(x_1 \ldots x_n, y_1 \ldots y_n).$$

For the case without Wiener integrals, $d = 0$, (7) can be verified directly as is done in Appendix B. The no longer elementary case $d = 1$ is briefly discussed in Appendix C.

For $d = 2$ and $d = 3$, it should be noted that a rigorous delta-function pair-potential has no physical effect[60], such that $\delta^{\beta}_{reg} \neq \delta$ for $\beta > 0$ is essential. However, the

density goes to infinity (at least for $d \leqslant 3$) more strongly than β^{-1} (at least for $d \geqslant 2$) such that an effect of the potential also in the delta-limit is plausible. The statements on the density result from (7) with $n = 1$ and the following bounds derived by other methods[11]

$$-\operatorname{Sup}\{0, -(2g)^{-1}\delta m^2\} \leqslant S(0,0) - G_o(0) \leqslant$$

$$\leqslant G_o(0) + g^{-1}\delta m^2 +$$

$$+ \{[G_o(0) + g^{-1}\delta m^2]^2 + G_o(0)^2\}^{1/2}$$

whereof, according to the remarks to (2.2), the first inequality is meaningful for $d \leqslant 2$ and the second for $d \leqslant 1$.

For $d = 4$ one expects also amplitude-and coupling-constant renormalization to be necessary; in (4'), (6') and (7') there are as many multipliers as necessary to take these effects into account. Thus it seems that the rules (4'), (6'), (7') provide at least a regularization[63] of EQFT (of charged particles). From this viewpoint it is significant that the result of Fisher and Ruelle[64] excludes rigorously the possibility of the existence of the thermodynamical limit for the QSM Hamiltonians one would associate with the EQFT to the Lagrangian density (3.12) since the potential would have to be essentially negative to approach singular attraction.

Moreover, the quantitative rules (4'), (6'), (7') suggest to speculate about counterparts in EQFT and, by inference, MQFT of collective phenomena in QSM as exhibited in condensed phases.

Appendix B. d = 0 : A Numerical Model

We shall write here the formulas of Chapter 2 to 4 for zero dimensions such that all Wiener integrals are absent, since then the exact solution is known and the KS and MM equations are elementary but not trivial although the model is.

(2.6) takes the form

$$S_x + g\,S_{xxy} - \alpha\,S_x = yS \tag{B.1a}$$

$$S_y + g\,S_{xyy} - \alpha\,S_y = xS \tag{B.1b}$$

for a function $S(x,y)$. All solutions of this system depend on $xy \equiv U$ only i.e. preserve gauge invariance of the first kind. With $S(x,y) = S(U)$(1) takes the form

$$S' + g(2\,S'' + US''') - \alpha\,S' = S. \tag{B.2}$$

The three solutions, all of which are nonalytic in g at g = 0, can be written

$$S_1(U) = C \int_{\rightsquigarrow} dz\ (1-\alpha-ig^{1/2}z)^{-1} \exp[-\tfrac{1}{2}z^2 + U(1-\alpha-ig^{1/2}z)^{-1}] \tag{B.3}$$

and

$$S_{2,3}(U) = C\ \mathrm{Re},\mathrm{Im} \oint dz\ (1-2-ig^{1/2}z)^{-1} \exp[-1/2z^2 + +U(1-\alpha-ig^{1/2}z)^{-1}]$$

Only $S_1(U)$ satisfies the analog of property (2.9),

$$\text{(B.4)} \qquad \sum_{i,j=1}^{k} \bar{C}_i C_j \; S\left((\bar{\alpha}_i + \bar{\beta}_j)\;(a_j + b_i)\right) > 0,$$

and possesses for $\alpha < 1$ an asymptotic expansion for $g \to 0$ from $|\arg g| < \pi$, while being analytic in the cut g-plane. It is also analytic in g and m^2 simultaneously in a certain domain. (3) is the analog of (3.3). Using (3.4) gives the analog of (3.8b)

$$\text{(B.5)} \qquad S_1(U) = \sum_{n=o}^{\infty} (n!)^{-1} U^n 2^{-n} \int_o^{\infty} \cdots \int ds_1 .. ds_n$$

$$. \exp[-\tfrac{1}{2}(s_1 + .. + s_n)] n(s_1 .. s_n)$$

where, explicitly,

$$\text{(B.6)} \qquad n(s_1 \ldots s_n) = n(s_1 + \ldots + s_n)$$

with

$$\text{(B.7)} \qquad n(s) = C \int dz\, (1-\alpha-ig^{1/2}z)^{-1} \exp[-\tfrac{1}{2}z^2 + \tfrac{1}{2}\alpha s +$$

$$+ \tfrac{1}{2}\, ig^{1/2} zs] =$$

$$= C'\, e^{s/2}\, \mathrm{Erfc}[2^{-3/2} g^{1/2} s + (2g)^{-1/2}(1-\alpha)]$$

where C resp. C' such that $n(0)=1$. The LE analogous to (3.9b) is again meaningless unless e.g. the lower limits of the t - integrations are raised and then converges exponentially. The MM equation shows explicitly that (6) holds and thus permits to sum up all terms on its right hand side to obtain the integral equation

$$\text{(B.8)} \qquad n(s) = \exp(\tfrac{1}{2}\alpha s - \tfrac{g}{8} s^2)[1 - \tfrac{gs}{4} \int_o^{\infty} dt\; n(t) \exp(-\tfrac{1}{2}t - \tfrac{gst}{4})]$$

which is solved by (7). The sum of terms converges absolutely since, summing absolutely and replacing $n(s)$ by $m(s)$, we obtain

$$\text{(B.9)} \quad m(s) = \exp\left(\frac{1}{2}\alpha s - (g/8)s^2\right)\left[1 + \frac{gs}{4}\int_0^\infty dt\, m(t)\exp\left(-\frac{1}{2}t\right)\right]$$

and with $n(t)$ for $m(t)$ the integral in (9) converges for (7). Using (6) we obtain for (5)

$$\text{(B.10)} \quad S_1(u) = 1 + u\int_0^\infty ds\, e^{-s/2}(2us)^{-1/2} I_1\left((2us)^{1/2}\right) n(s)$$

which converges absolutely for all u and is the solution[65] (3) of (2). Thus, the intermediate use of the formal LE to arrive at the MM equation has introduced no error and $n(s)$ is positive as we expect it to be also for $d > 0$.

The construction of a solution of (8) by iteration from the inhomogeneous term is certainly possible if also the iteration solution of (9) converges, which is the case if and only if $\alpha < 1$. The natural value for α, according to (2.2), may be taken as $2g$, and then g should not be too large. To consider instead the convergence of the iteration solution of (8) presupposes that the cancellations between the terms of alternating algebraic signs are brought to bear since if $\alpha > 1$ not all iterative approximations to $n(s)$, starting from the inhomogeneous term, are everywhere non-negative. Even if this cancellation is observed, the iteration solution of (8) does not converge for α too large. However, it does converge for α smaller than a g - dependent bound greater than one and gives

the for these α unique solution of (8). We have not shown that (8) does not possess homogeneous solutions for α large enough. (It should be stressed that the boundary conditions on the determination of $n(s)$ are only that it be non-negative and that the integrals in (8), and in (10) for an infinitesimal environment of the origin, converge.)

The KS equation does not make it manifest that $n(s_1 \ldots s_n)$ obeys (6). However, since the MM equation is a consequence of the KS equation and implies (6), we will first for simplicity consider only the solutions of this form. Then again the summation of all terms on the right hand side of the KS equation can be performed and leads to

$$\text{(B.11)} \qquad n(s + s') = \exp\left(\frac{1}{2}\alpha s - \frac{g}{8} s^2 - \frac{g}{4} ss'\right) \cdot$$

$$\cdot [n(s') - \frac{gs}{4} \int_0^\infty dt\, n(t + s') \exp(-\frac{t}{2} - \frac{g}{4} st)].$$

This implies

$$n''(s) + 4^{-1}(-2 - 2a + gs)n'(s) + 8^{-1}(2\alpha - gs)n(s) = 0$$

whereof the solution besides (7) is $\exp(s/2)$ which does not solve (11) while (7) does. Thus, (11) has only the solution (7).

The discussion of the convergence of an iteration solution of the KS equation is lengthier since, e.g. starting the iteration from $n(\emptyset) = 1$ all approximations give functions that depend only on the sum of their arguments but no longer one universal function for all n in a given step of iteration. However, in this special case the discussion can be reduced to the former one of the MM equation and the final result is the same.

The discussion of the MM and KS equations for reduced functionals is more complicated since the reduced functions depend no longer on the sum of arguments only, and has not been done.

The analog of the theory described by (3.11) has instead of (1)

(B.12) $$S_x + g\, S_{xxx} - \alpha\, S_x = xS$$

The change in the KS and MM equations is the same as that described for (3.11), and the discussion of (12)[66] leads in every detail to similar results as were obtained for (1).

The model (1) stands in the relation described in appendix A to the model given by

$$H = 1/2\ V\ a^+ a^+ aa,\ [a, a^+] = 1$$

The "n-particle distribution function" is

$$\rho_n(\beta) = [\mathrm{Tr}\ \exp(-\beta H + \beta\mu N)]^{-1} \mathrm{Tr}\{(a^+)^n a^n \exp(-\beta H + \beta\mu N)\}$$

where $N = a^+ a$. The generating function is

$$S_\beta(x) = \sum_{n=o}^{\infty} (n!)^{-2} x^n \rho_n(\beta) =$$

$$= C\ \mathrm{Tr}\{\exp\ (-\beta H + \beta\mu N) e^{xa^+} e^{a}\}$$

which is easily calculated to be

$$S_\beta(x) = C' \int dz\{1-\exp[\beta\mu+\frac{1}{2}\beta V+iz(\beta V)^{1/2}]\}^{-1} .$$

$$.\exp\{-\frac{1}{2}z^2+x[-1+\exp(-\beta\mu-1/2\beta V-iz(\beta V)^{1/2}]^{-1}\} .$$

The substitution corresponding to (A.4), (A.6) and (A.7)

$$V = \frac{1}{4}\beta g,\ x = (1/2)\beta U,\ \mu = -1/2 + \alpha/2$$

leads, with $C' \sim \beta$, for $\beta \to 0$ to (3).

Appendix C. d = 1: The Anharmonic Oscillator

For d = 1, (2.1) becomes

(C.1) $$L = \dot{B}^+\dot{B} - m^2 B^+B - 1/2\, g(B^+B)^2 + \alpha\, B^+B.$$

With $B = r \exp(i\phi)$, $B^+ = r\ \mathrm{esp}(-i\phi)$ the Hamiltonian becomes

(C.2) $$H = 4^{-1}[-\partial^2/(\partial r)^2 - r^{-1}\partial/\partial r - r^{-2}\partial^2/(\partial\phi)^2 + 4(m^2-\alpha)r^2 + 2gr^4]$$

which commutes with $-i\partial/\partial\phi$. The Green's functions

$$< TB(t_i)..B(t_n)B^+(t_1')..B^+(t_n') >$$

can be continued analytically as described in chapter 1 and an EQFT be based on them as described in chapter 2.

The Hamiltonian (2.10) is a typical field theoretical one and as such beset with the familiar features concomitant with an infinite volume. There are no ultraviolet divergences, however. The energy spectrum is now continuous except for the vacuum state which may be separated by a gap[67] from the onset of the continuum. The Green's functions are the vacuum expectation values of the equal-time field-operator products. The theory differs from a two-dimensional MQFT by not possessing relativistic invariance, but is likely to admit a particle interpretation as it does for g = 0 with, however, an unusual energy-momentum relation. The eigenvalues and selection rules of the two-dimensional oscillator (2) manifest themselves in the

familiar asymptotic decrease (1.3) of the equal-time vacuum expectation values for large distances. Gauge invariance of the first kind is, of course, not broken.

The volume would be finite, with periodic boundary conditions, if we had taken the finite-temperature expectation[28] of the time-ordered operator product instead of the ground state expectation. At least in the interaction-free case $g = 0$, the energy spectrum is now also discrete since it is then simply related to the discrete momentum spectrum.

All formulae of chapters 3 and 4 hold with Wiener trajectories in one-dimensional space resp. with one-dimensional blobs. As all diagrams are now convergent, we need not introduce reduced functionals and discuss the KS and MM equations directly, following closely Ginibre.[17] The linear vector space of sequences of Wiener-integrable functionals $n(\omega_1 \ldots \omega_n)$, $n = 1, 2, \ldots$ is made a Banach space by choosing as norm

$$\| N \| = \operatorname*{Sup}_{n} \ \operatorname*{ess.\,sup}_{\omega_1 \ldots \omega_n} |n(\omega_1 \ldots \omega_n)| g(\omega_1)^{-1} g(\omega_n)^{-1} \tag{C.3}$$

with a functional $g(\omega)$ to be suitably chosen, and completing. The operator Op of (4.6) has for the KS and MM equation a norm bounded by $c < 1$ if

$$g(\omega) > c \exp \left[1/2\ \alpha\ s + \int_0^\infty t^{-1} dt\ e^{-m^2 t/2} \int dz \int P^t_{zz}(d\bar{\omega}) K(\omega,\bar{\omega}) g(\bar{\omega})\right] \tag{C.4}$$

where we have used that $V_{ij} \geqslant 0$ and, in the MM case, (5.9). We may restrict ourselves to translation-invariant $g(\omega)$. Then, with (4.1),

$$\int dz \int P^t_{zz}(d\bar{\omega})K(\omega,\bar{\omega})g(\bar{\omega}) =$$

$$= \int P^t_{oo}\ (d\bar{\omega})g(\bar{\omega}) \int dz\ K\ (\omega,\bar{\omega} + z) <$$

$$< 4^{-1}gst \int P^t_{oo}\ (d\bar{\omega})\ g\ (\bar{\omega}).$$

Thus, setting

$$\int_o^\infty dt\ e^{-m^2 t/2} \int P^t_{oo}\ (d\bar{\omega})\ g\ (\bar{\omega}) = a$$

we satisfy (4) by choosing

$$g\ (\omega) = c\ \exp[4^{-1}(2\alpha s + g\ s\ a)]$$

from which follows

$$a = c\ (m^2 - \alpha - ag/2)^{-1/2}$$

which can be solved for a with c < 1 provided

$$m^2 - \alpha > 3\ (g/4)^{2/3} \tag{C.5}$$

i.e. the anharmonicity should not be too large for the iteration solution of (4.6) to converge to the then unique solution according to this estimate. Especially, $\alpha < m^2$ would be required which would correspond to $\alpha < 1$ from (B.9). It is likely that this restriction is due to the present crude

estimate only since it did not apply for (B.8), although on the basis of the results of appendix B one would not expect the iteration from the N_o in (4.6) to converge for arbitrarily large g and α.

For complex g the iteration solution converges uniformly if Re $g > 0$ and in (5) g is replaced by $|g|$, and therefore this solution is an analytic function of g in the open semicircle.[68] It is in addition analytic in $m^2-\alpha$ provided

$$\mathrm{Re}\,(m^2-\alpha) > 3\,(|g|/4)^{2/3},\ \mathrm{Re}\,g > 0,$$

where it should be kept in mind that in view of (2.2) it is natural to have α depend on g.

The QSM model corresponding to this EQFT in the sense of appendix A is the quantum gas of nonrelativistic neutral scalar bosons in one dimension with repulsive delta-function pair potential. This model has been solved exactly by Lieb and Liniger[69], who observed nonanalytic behaviour of e.g. the ground state energy at $g = 0$ and no phase transition in accordance with Landau and Lifshitz.[70] It is interesting that already in this case, which is free of ultraviolet divergences, (A.6) should not be taken as it stands since we may set $\delta^{\beta}_{reg} = \delta$ already for $\beta > 0$. This will be cleared up by "QSM regularization" of (5.7a) in our next paper.

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20. See ref. 21 for a precise statement of the assumption.

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26. If the formal invariance of (2.1) under gauge transformation of the first kind is not violated by the vacuum, Green's functions with $m \neq n$ are zero. No assumption to this effect is needed in this chapter.

27. In this and other formulas that do not make reference to (2.1)or to canonical commutation relations, the field operators should be the renormalized ones whenever the amplitude renormalization is not finite; then the formulae remain valid also for $d = 4$.

28. Integration with the Green's function that satisfies periodic boundary conditions in one coordinate would lead to finite-temperature Green's functions, see e.g. A.A. Abrikosov, L.P. Gorkov, I. E. Dzyaloshinski,

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29. By special choices of the C_i and J_i one can derive from (9) inequalities that resemble the positive-definiteness conditions for Wightman functions[19]. However, there is no simple interrelation since the present inequalities do not require the MQFT metric to be positive definite. Because of their generality, they might hold even in axiomatically introduced EQFT but we have no proof of this.

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35. This form will be seen to exclude Green's functions that do not obey gauge invariance of the first kind. To admit the nonvanishing of those, one would in (1) have to introduce the fictitious symmetry-violating terms used by Bogoliubov (Rochester Conference, 1957). We will not pursue this possibility here. See also the discussion of symmetry-breaking in chapter 4.

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45. The same holds if in (1) g is chosen negative instead of positive.

46. E.g., we prove in the next paper that for $d \geqq 2$ V_{ii} is almost everywhere infinite on Wiener space.

47. Such regularization is actually gauge invariant in QED.

48. The corresponding redundancy in QSM, ref. 17, is slight.

49. $f(z)$ is a functional of the Wiener process $z(\tau)$, specifically, the occupation-time distribution.

50. The integral (4.5a) will exist (i.e., not be "ultraviolet divergent") only if $F(b)$ vanishes sufficiently strongly for $\| b \| \to 0$.

51. The divergence mentioned in ref. 46 implies that (in blob measure) almost no blob function is square integrable.

52. This objection was raised by H. Stumpf at the Seminar on Unified Theories of Elementary Particles (Feldafing, July 1965) where the material of this paper was presented.

53. Note that a formulation of renormalized EQFT by the usual coupled integral equations for Green's functions need not define the theory uniquely while the KS and MM equations may incorporate the missing boundary conditions. This is in fact so for the model discussed in appendix B.

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56. The rigorous results for infinite volume (see: D. Ruelle, in "Lectures in Theoretical Physics", vol. VI (University of Colorado Press, Boulder, 1964), p. 37) cover only the domain of sufficiently small activity i.e. at most the gaseous phase.

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61. The definition (3.7) corresponds to particle mass $M = 1$ instead of 2 as in G(2.3).

62. This will be required, due to tadpole divergence, for the limit (7) to exist even if δ^{β}_{reg} is not made to become a delta function, cp. chapter 4.

63. This may be regarded even as a regularization of MQFT if one uses a δ^{β}_{reg} that is a delta function in one coordinate direction, which would correspond to the original, MQFT, time. Hoever, MQFT not

regularized further would still be ultraviolet divergent, see ref. 62.

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67. If $g = 0$ and $m^2 = 0$ there is no gap since the theory is then the usual one of free nonrelativitistic particles. If $g = 0$ and $m^2 > 0$ there is a gap.

68. The method of ref.11, which does not rely on iteration, can be used to prove that the EQFT functions in finite one-dimensional volume with periodic boundary conditions, see ref. 28, are analytic in g in the right g half-plane, and that the perturbation expansion is an asymptotic one.

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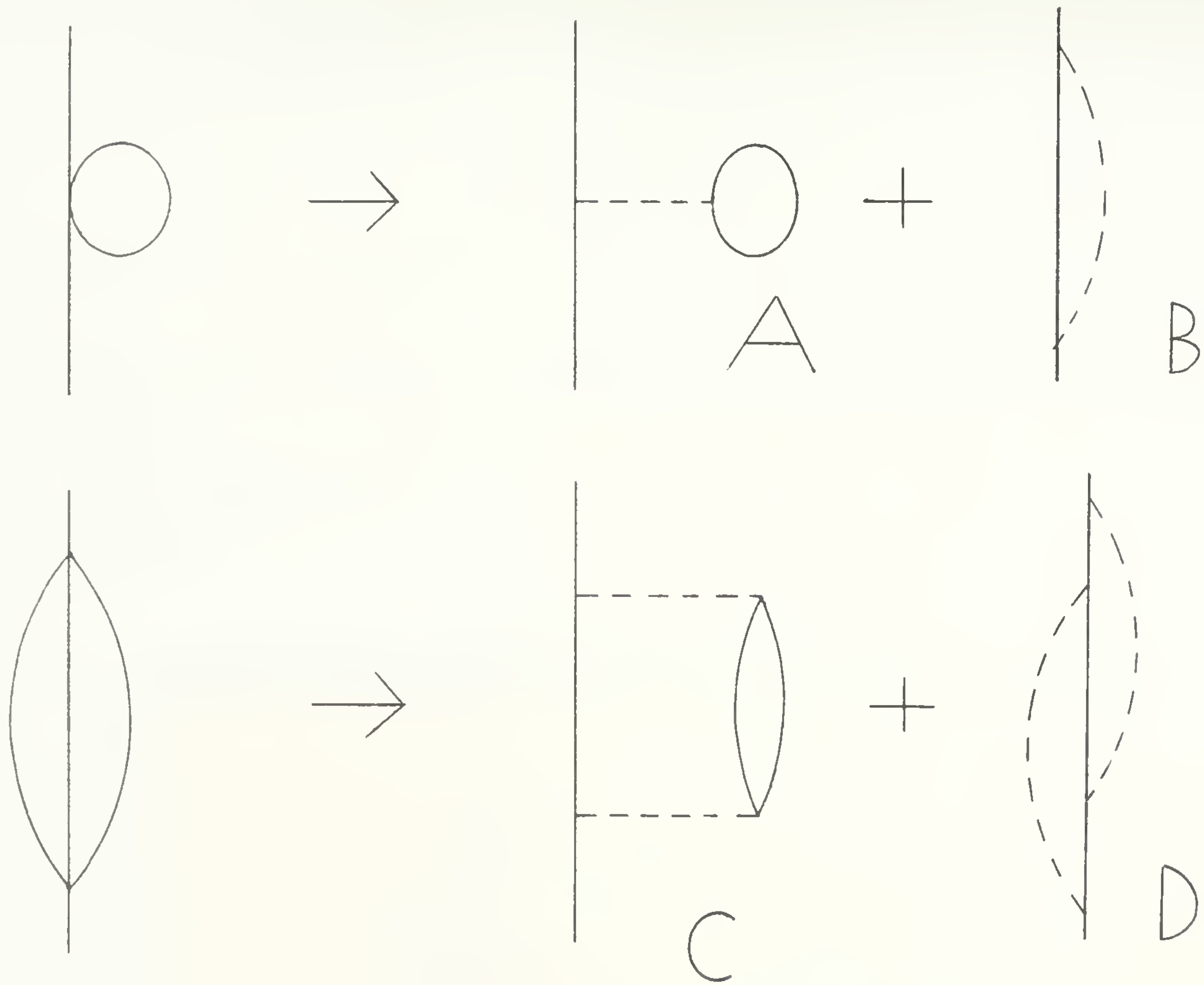

Fig. 1   Primitively divergent diagrams for $d. = 3$